

# Optimal energy storing and selling in continuous time stochastic multi-battery setting

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## Abstract

We consider the problem of optimal energy storing and dispatching for a grid-connected energy producer with battery energy storage system (BESS). We suggest an optimal operating algorithm that helps to increase the income given limited storage capacity and unpredictable fluctuations of the production rate and selling prices. In addition, the operating algorithm takes into account preferable regimes for charging and discharging the batteries; this can help to prolong the battery life. The problem is solved as a special stochastic control problem in a domain where the state processes have to remain in a domain but are allowed to run on the domain boundary. An equation for the optimal value function is derived in continuous time setting. Numerical methods based on duality and pathwise optimization for estimation of the optimal solution are suggested.

**Key words:** energy storage, energy dispatching, battery energy storage system (BESS), stochastic control, optimal control

**2010 Mathematics Subject Classification:** 91B70, 93E20

## 1 Introduction

Recent wide expansion of new energy technologies and growth of the number of small and medium companies producing and selling energy triggered need for new types of operating algorithms to

ensure sustainable and reliable production process and maximization of the profit. An important feature of energy production based on the renewable sources is that unpredictable fluctuations of the production rate can be significant. To compensate these fluctuation and ensure more stable output, the energy has to be stored. Typically, it is necessary to consider a storage consisting of several separate battery units that have to be regularly charged and discharged i.e. a battery energy storage system (BESS).

In addition, the selling price for energy is also fluctuating unpredictably. Because of these fluctuations, it is beneficial to store energy when the price is low and dispatch the energy when the price is high. This helps to generate extra income and increase profitability.

The fluctuations of the production rate represent a mixture of relatively regular predictable components such as night interruptions for solar energy and tide cycles for wave energy, and of irregular unpredictable components such as fluctuations caused by weather conditions for solar and wind energy; see e.g. [8, 12, 17]. The fluctuations of the selling price also represent a mixture of relatively regular and predictable components such as switching between day and night prices and irregular unpredictable components caused by unpredictable market movements. Overall, the fluctuations of both production rate and selling prices are non-predictable in the sense that their forecast without error is impossible; see e.g. [19].

Therefore, the power producers need optimal strategies for energy storing and selling that help to reduce the impact of unpredictability of production rate and market prices. The problem appears to be a control problem under uncertainty, that, in the case of renewable energy, is defined by non-controlled external factors such as the weather. These questions are important for applications in energy sector, particularly for small and medium producers of renewable energy. The problem was studied intensively; see e.g. [18, 21, 24, 25, 28, 29].

The present paper suggests a comprehensive and yet compact dynamic model of energy dispatching and storage for a multi-battery setting. This model represents a further development of models suggested in [18, 25, 28, 29]. The main novelty is that our setting takes a special feature of the energy trading: the storage is based on batteries requiring certain regimes of charging and discharging to prolong the battery life; see e.g. [20, 23, 25] and the bibliography therein. Given that the batteries are expensive, this is a significant factor in decision making. To address this, we considered an extended model where cumulative moving averages were included.

The main focus of the paper is an optimization model for energy storing and dispatching processes; however, we suggested some ways of solution of optimal control problems arising for this model. For analysis, we use the approach from [13] based on dynamic programming.

The problem was solved as a special stochastic control problem in domain where state processes are allowed to go on and off the boundary of the admissible domain as well as stay on the boundary. This is an unusual setting for stochastic optimal control, where processes with reflection from the boundary or being killed at the boundary are more common. We derived the equation for the optimal value of the problem in a form of a Hamilton-Jacobi-Bellman (HJB) equation, and obtained some existence results. For a large number of factors arising in a multi-battery setting, the state space dimension for the HJB equation could be high. This means that numerical solution could be challenging. To address this, we suggested an alternative approach based on duality and pathwise optimization, in the spirit of [2, 3, 4, 6, 9, 10, 11, 15, 26]. In the framework of this method, the optimal value function can be calculated using Monte-Carlos simulation of the Lagrange terms and pathwise deterministic optimization in a class of non-adapted processes. Even if this does not lead to an optimal strategy immediately, it gives an opportunity to estimate how far from optimal is the performance of a particular strategy, for instance, such as suggested in [18].

The rest of the paper is organized as follows. In Section 2, we describe the basic model setting with a single battery. In Section 3, we introduce a multi-battery setting and discuss optimization of battery regimes. In Section 4, we derive Hamilton-Jacobi-Bellman equations describing the optimal value functions. In Section 5, we suggest a duality based method of numerical calculations of the optimal value functions. Section 6 contains the proofs. Section 7 offers some discussion and concluding remarks.

## 2 Problem setting: the basic model

Assume that  $p(t)$  is a stochastic process representing the current rate of production of the energy, and that  $S(t)$  is a stochastic process representing the current price of the energy unit,  $t \in [0, T]$ .

Assume that  $p(t)$  is a stochastic process representing the current rate of production of the energy, and that  $S(t)$  is a stochastic process representing the current price of the energy unit,  $t \in [0, T]$ , where  $T > 0$  is a given terminal time. These processes describe random evolutions,

since each may depend on unpredictable factors (weather, market conditions). The decision that has to be made by the producer can be represented as separation

$$p(t) = s(t) + u(t),$$

where  $s(t)$  is the rate of the selling of energy, and  $u(t)$  is the rate of depositing the energy into a storage (a battery). The case where  $u(t) < 0$  is not excluded; in this case, this is the rate of withdrawing the energy from the storage. We assume that  $u(t) \in [-L, \min(p(t), L)]$ , where  $L > 0$  is given; the processes  $p, s$ , and  $S$ , are non-negative. In addition, we consider restrictions on the storage capacity such that

$$y(t) = y(0) + \int_0^t u(s)ds \in [0, C], \quad (1)$$

where  $C > 0$  is given. The process  $y(t)$  represents the quantity of the energy currently stored in the battery;  $C$  represents the battery capacity.

In this setting, the process  $u(t)$  is a control to be selected using the historical observations of  $(p(t), S(t))$  and as well as other currently available information such as the weather data or currency exchange rate.

Let  $T > 0$  be a given terminal time. The monetary value of the output of a particular strategy  $u$  at time  $T$  can be represented as

$$F(u) = F_s(u) + S(T)y(T).$$

Here

$$F_s(u) = \int_0^T (p(t) - u(t))S(t)dt$$

is the value representing the total earning from the selling during the time period  $[0, T]$ . The value  $S(T)y(T)$  represents the market value of the stored energy. The goal is to maximize the expectation of  $F$  over  $u$  given a probability distribution describing the current hypothesis on  $(p, S)$  and on other factors.

Let us give a more accurate description of information available for the decision making. We assume that the expectation is defined on a standard probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , where  $\Omega$  is a set of elementary events,  $\mathcal{F}$  is a complete  $\sigma$ -algebra of events, and  $\mathbf{P}$  is a probability measure.

Let  $\{\mathcal{F}_t\}_{t \geq 0}$  be the filtration generated by current observations and augmented by all the  $\mathbf{P}$ -null sets in  $\mathcal{F}$ . The processes  $p$ ,  $S$ ,  $s$ , and  $u$  have to be  $\mathcal{F}_t$ -adapted; however, this filtration may include information generated by other processes such as the weather etc.

Let  $\mathcal{U}$  be the class of processes  $u(t)$  that are progressively measurable with respect to the filtration  $\mathcal{F}_t$  and such that  $u(t) \in [-L, \min(p(t), L)]$  for all  $t$ .

In addition, we consider restrictions on the storage capacity such that

$$y(t) \in [0, C],$$

where  $C > 0$  is given. The process  $y(t)$  represent the quantity of the energy currently stored in the battery;  $C$  represents the battery capacity.

Let  $\mathcal{U}_C$  be the class of processes  $u \in \mathcal{U}$  that are progressively measurable with respect to the filtration  $\mathcal{F}_t$  and such that  $y(t) \in [0, C]$  for all  $t$ .

The following stochastic optimal control problem arises: for given  $t < T$  and  $y_0 \in [0, C]$ ,

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E} \left[ \int_t^T (p(s) - u(s))S(s)ds + y(T)S(T) \right] \quad \text{over } u(\cdot) \in \mathcal{U}_C \\ \text{subject to} \quad & \frac{dy}{ds}(s) = u(s), \quad y(t) = y_0, \quad y(t) \in [0, C]. \end{aligned} \quad (2)$$

It can be noted that admissible state processes  $y$  are allowed to go on and off the boundary of the admissible domain as well as stay on the boundary. This is an unusual setting for stochastic optimal control, where processes with reflection from the boundary or being killed at the boundary are usually considered.

**Theorem 2.1** *Problem (2) is equivalent to the problem*

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E} \left[ \int_t^T \mathbb{I}_{\{y(s) \geq 0\}} (p(s) - u(s))S(s)dt + y(T)S(T) \right] \quad \text{over } u(\cdot) \in \mathcal{U} \\ \text{subject to} \quad & \frac{dy}{ds}(s) = u(s)\mathbb{I}_{\{y(s) \leq C\}}, \quad y(t) = y_0. \end{aligned} \quad (3)$$

The proofs of all theorems are given in the Appendix below.

## 2.1 Diffusion case

Let us assume that  $(p(t), S(t))$  is a part of a stochastic Markov diffusion process.

Let  $w(\cdot)$  be a standard  $n$ -dimensional Wiener process,  $n \geq 2$ . Let  $\bar{g} : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$  and  $\bar{\beta} : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$  be some continuous functions such that  $|\bar{g}(x, t)| + |\bar{\beta}(x, t)| \leq \text{const}(|x| + 1)$  and  $|\partial \bar{g}(x, t)/\partial x| + |\partial \bar{\beta}(x, u, t)/\partial x| \leq \text{const}$  for all  $x, u, t$ .

Starting from now, we assume that  $\mathcal{F}_t$  is the filtration generated by  $w(t)$ ,

$$p(t) = e^{\bar{x}_1(t)}, \quad S(t) = e^{\bar{x}_2(t)}, \quad (4)$$

where  $\bar{x}(t) = (\bar{x}_1(t), \dots, \bar{x}_n(t))^\top$  is a stochastic diffusion process evolving as

$$d\bar{x}(t) = \bar{g}(\bar{x}(t), t)dt + \bar{\beta}(\bar{x}(t), t)dw(t). \quad (5)$$

The components  $\{\bar{x}_k(t)\}_{k>2}$  represent currently available and unpredictable information (other than  $(p(t), S(t))$ ) such as the weather data or a currency exchange rate.

Equations (4)-(5) define a stochastic evolution model for the process  $(p(t), s(t))$ . Calibration of the parameters for these equations is a complicated task involving statistical inference and forecasting methods; this task is .

Matching of the definitions shows that problem (3) can be rewritten in the form of a problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E} \left[ \int_0^T h(x(t), u(t), t)dt + \Phi(x(T)) \right] \quad \text{over} \quad u(\cdot) \in \mathcal{U}_C \\ \text{subject to} \quad & dx(t) = g(x(t), u(t), t)dt + \beta(x(t), t)dw(t), \quad x(0) = x_0. \end{aligned} \quad (6)$$

Here

$$\begin{aligned} x_0 &= \begin{pmatrix} \bar{x}_0 \\ y_0 \end{pmatrix} \in \mathbf{R}^{n+1}, \quad x(t) = \begin{pmatrix} \bar{x}(t) \\ y(t) \end{pmatrix}, \quad g(x, u, t) = \begin{pmatrix} \bar{g}(x_1, \dots, x_n, t) \\ f(x_1, \dots, x_n, u, t) \end{pmatrix}, \\ \beta(x, u, t) &= \begin{pmatrix} \bar{\beta}(x_1, \dots, x_n, t) \\ 0_{\mathbf{R}^{1 \times n}} \end{pmatrix}, \quad f(x_1, \dots, x_n, u, t) = u\mathbb{I}_{\{x_{n+1} \leq C\}}, \\ h(x, u, t) &= \mathbb{I}_{\{x_{n+1} \geq 0\}}(e^{x_1} - u)e^{x_2}, \quad \Phi(x) = e^{x_2}x_{n+1}, \end{aligned}$$

where  $x = (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1}$ ,  $t \in [0, T]$ .

### 3 Multi-battery model with optimization of the battery regimes

Consider now situation where the storage consist of several separate units with some preferable regime of their operations. More precisely, we consider a setting where the energy is stored

in several batteries. To take this into account, we extend the model introduced above as the following.

We consider processes

$$\begin{aligned} y(t) &= (y_1(t), \dots, y_m(t))^\top, \quad u(t) = (u_1(t), \dots, u_m(t))^\top, \\ y_i(t) &= y_{0i} + \int_0^t u_i(s) ds, \quad y_i(t) \in [0, C]. \end{aligned}$$

The technology reasons suggest certain regimes for charging and discharging batteries used to storage energy by the producer. Given that the batteries are expensive, this could be a significant factor in decision making. It is known that too fast charging and discharging may lead to shortened battery life [20]. This can be controlled by using  $L$  in our setting. In addition, a deep discharge may also have negative effect [20]. To take this into account, we may incorporate additional task of maximization of

$$\mathbf{E} \int_0^T \widehat{\phi}(y(t), u(t)) dt \quad (7)$$

where  $\widehat{\phi} : \mathbf{R}^m \times \mathbf{R}^m \rightarrow (-\infty, 0)$  is a function that achieves minimum on the boundary of the domain  $[0, C]^m \times [-L, L]$  (or on a selected part of the boundary). For example, one may select  $\widehat{\phi}(u, y) = -\prod_{i=1}^m \varphi_i^u(u_i) \varphi_i^y(y_i)$ , where  $\varphi_i^u : [-L, L] \rightarrow (0, +\infty)$  and  $\varphi_i^y : [0, C] \rightarrow (0, +\infty)$  are some  $U$ -shaped convex functions.

### 3.1 Preferences using cumulative moving averages

It appears that some important criterions cannot be covered by integrals of the functions of the current state  $y(t)$ . In some cases, it could be reasonable to use criterions of quality of regimes that involve average paths, i.e. cumulative moving averages

$$\bar{y}_i(t) \triangleq \frac{1}{t} \int_0^t y_i(s) ds.$$

This can be described as via minimization of the expectation

$$\mathbf{E} \int_0^T \bar{\phi}(y(t), \bar{y}(t)) dt, \quad (8)$$

where  $\bar{\phi} : \mathbf{R}^m \times \mathbf{R}^m \times (0, T] \rightarrow \mathbf{R}$  is a function describing the agent's preferences.

For instance, a preference that the charging processes are oscillating with a similar rate for all batteries can be taken into account with

$$\bar{\phi}(y(t), \bar{y}(t)) = - \sum_{i,j=1}^m (Y_i(t) - Y_j(t))^2,$$

where  $Y_i = y_i(t) - \bar{y}_j(t)$ .

It can be noted that this performance criterion cannot be captured via maximization of (7).

Alternatively, one may prefer to have all batteries changing their charge with the same rate. For this, one can use

$$\bar{\phi}(y(t), \bar{y}(t)) = - \sum_{i,j=1}^m (\bar{y}_i(t) - \bar{y}_j(t))^2 dt.$$

Using of moving averages in the criterions helps to control regularity of circles of particular batteries via maximization of (8) with

$$\bar{\phi}(y(t), \bar{y}(t)) = \Gamma \sum_{i,j=1}^m (y_i(t) - \bar{y}_j(t))^2. \quad (9)$$

Here  $\Gamma \in \mathbf{R}$  is a constant that defines the weight of criterion (11) for decision making.

Clearly, maximum of the expectation in (8) with  $\Gamma < 0$  is achieved for the batteries with constant levels of energy stored.

Let us demonstrate the impact of maximization of (8) given (9) with  $\Gamma > 0$ . Consider the following problem:

$$\text{Maximize} \quad \int_0^T (v(t) - \bar{v}(t))^2 dt, \quad (10)$$

$$\text{subject to} \quad v(t) \in [0, C], \quad \frac{dv}{dt}(t) \in [-L, L], \quad (11)$$

where

$$\bar{v}(t) \triangleq \frac{1}{t} \int_0^t v(s) ds.$$

These solutions have to deviate from their historical mean as much as possible. In fact, problem (11) can be represented as a linear quadratic problem

$$\begin{aligned} & \text{Maximize} \quad \int_0^T (v(t) - \bar{v}(t))^2 dt \quad \text{over} \quad v(\cdot), \\ & \text{subject to} \quad d\bar{v}(t)/dt = -t^{-2}\bar{v}(t) + t^{-1}v(t), \quad v(t) \in [0, C], \quad \frac{dv}{dt}(t) \in [-L, L]. \end{aligned}$$



It appears that this criterion lead to periodic regimes with stable oscillations. To show this, we did the following experiments. We created a set of discrete time paths path  $u = (u^{(1)}, \dots, u^{(N)})$  such that  $u^{(k)} = \pm C$ , where  $N$  is the time discretization parameter. This set of paths was created using Monte-Carlo simulation of binary vectors with independent components. The corresponding process  $v(t)$  was replaced by the vector  $v = (v^{(1)}, \dots, v^{(N)})$  such that  $v^{(j)} = v^{(1)} + \sum_{d=1}^j v^{(d)} \Delta t$ , where  $\Delta t = T/N$ . The approximation of the optimal path was identified as the path with the minimal value of

$$\sum_{j=1}^n (v^{(j)} - \bar{v}^{(j)})^2$$

where

$$\bar{v}^{(j)} = j^{-1} \sum_{d=1}^j v^{(d)}.$$

The approximation of the path among Monte-Carlo simulated 2,000 paths is presented by Figure 3.1. For this experiment, we used  $T = 1$ ,  $C = 1$ ,  $L = 100$ ,  $n = 1000$ .

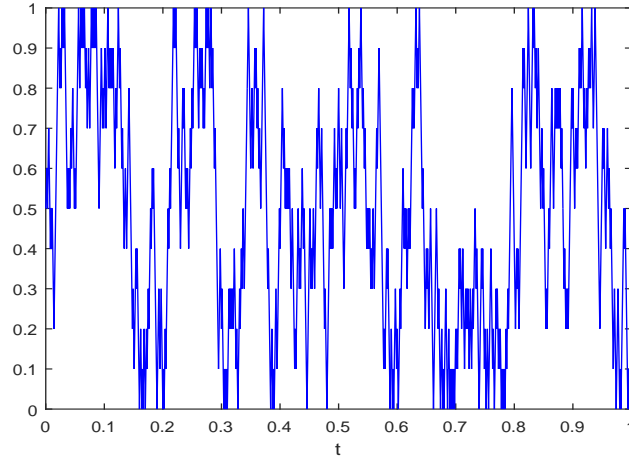


Figure 3.1: Approximation of optimal solution of problem (11)

### 3.2 Optimal control setting for the multi-battery model

Let

$$\phi(y(t), \bar{y}(t), u(t), t) = \hat{\phi}(y(t), u(t)) + \bar{\phi}(y(t), \bar{y}(t), t),$$

where  $\hat{\phi}$  and  $\bar{\phi}$  are selected such as described above, with the purpose to take into account the preferences for the battery regimes.

The following stochastic optimal control problem arises: for given  $t < T$  and  $y_0 \in [0, C]^m \subset \mathbf{R}^m$ ,

$$\begin{aligned}
& \text{Maximize} \\
& \mathbf{E} \int_t^T \left[ \left( p(s) - \sum_{i=1}^m u_i(s) \right) S(s) + \phi(y(s), \bar{y}(s), u(s), s) \right] ds + \mathbf{E} S(T) \sum_{i=1}^m y_i(T) \\
& \text{over } u(\cdot) \in \mathcal{U}_C, \\
& \text{subject to } \frac{dy_i}{ds}(s) = u_i(s), \quad y_i(t) \in [0, C], \quad i = 1, \dots, m, \quad y(t) = y_0.
\end{aligned} \tag{12}$$

**Theorem 3.1** *Problem (12) is equivalent to the problem*

$$\begin{aligned}
& \text{Maximize} \\
& \mathbf{E} \int_t^T \left[ \mathbb{I}_{\{\min_{i=1, \dots, m} y_i(s) \geq 0\}} \left( p(s) - \sum_{i=1}^m u_i(s) \right) S(s) + \phi(y(t), \bar{y}(t), u(t), t) dt \right] dt \\
& \quad \quad \quad + \mathbf{E} S(T) \sum_{i=1}^m y_i(T) \\
& \text{over } u(\cdot) \in \mathcal{U}, \\
& \text{subject to } \frac{dy_i}{ds}(s) = u_i(s) \mathbb{I}_{\{y_i(s) \leq C\}}, \quad i = 1, \dots, m, \quad y(t) = y_0.
\end{aligned} \tag{13}$$

### 3.3 Diffusion multi-battery model

Assume that  $(p(t), S(t))$  is a stochastic diffusion process  $\bar{x}(t)$  defined as (4)-(5), and that  $\mathcal{F}_t$  is the filtration generated by  $w(t)$ . Matching of the definitions shows that problem (13) is equivalent to the problem

$$\begin{aligned}
& \text{Maximize} \quad \mathbf{E} \left[ \int_0^T h(x(t), u(t), t) dt + \Phi(x(T)) \right] \quad \text{over } u(\cdot) \in \mathcal{U}_C \\
& \text{subject to} \quad dx(t) = g(x(t), u(t), t) dt + \beta(x(t), t) dw(t), \quad x(0) = x_0.
\end{aligned} \tag{14}$$

Here  $t \in [0, T]$ ,  $x(t) = (x_1(t), \dots, x_{n+2m}(t))^\top$ ,  $e^{x_1(t)} = e^{\bar{x}_1(t)} = p(t)$ ,  $e^{x_2(t)} = e^{\bar{x}_2(t)} = S(t)$ ,  $(x_{n+1}(t), \dots, x_{n+m}(t))^\top = (y_1(t), \dots, y_m(t))^\top$ ,  $(x_{n+m+1}(t), \dots, x_{n+2m}(t))^\top = (\bar{y}_1(t), \dots, \bar{y}_m(t))^\top$ ,  $f =$

$$(f_1(t), \dots, f_{2m})^\top,$$

$$x_0 = \begin{pmatrix} \bar{x}_0 \\ y_0 \\ \bar{y}_0 \end{pmatrix} \in \mathbf{R}^{n+2m}, \quad x(t) = \begin{pmatrix} \bar{x}(t) \\ y(t) \\ \bar{y}(t) \end{pmatrix},$$

$$g(x, u, t) = \begin{pmatrix} \bar{g}(x_1, \dots, x_n, t) \\ f(x_1, \dots, x_n, u, t) \end{pmatrix}, \quad \beta(x, u, t) = \begin{pmatrix} \bar{\beta}(x_1, \dots, x_n, t) \\ 0_{\mathbf{R}^{1 \times n}} \end{pmatrix},$$

$$h(x, u, t) = \mathbb{I}_{\{x_{n+i} \geq 0, i=1, \dots, m\}} \left( e^{x_1} - \sum_{i=1}^m u_i \right) e^{x_2} + \phi(\{x_{n+i}\}_{i=1}^m, \{x_{n+m+i}\}_{i=1}^m, u, t),$$

$$\Phi(x) = e^{x_2} \sum_{i=n+1}^{n+m} x_i,$$

where

$$\begin{aligned} f_i(x_{n+i}, \dots, x_{n+m}, u, t) &= u_i \mathbb{I}_{\{x_{n+i} \leq C\}}, & i &= 1, \dots, m, \\ f_i(x_{n+m+i}, \dots, x_{n+2m}, u, t) &= -t^{-2} x_{n+i} + t^{-1} x_{n+i}, & i &= m+1, \dots, 2m. \end{aligned}$$

## 4 The dynamic programming approach

The state equations for problems (6) and (14) are degenerate which make them difficult for analysis. In addition, they have discontinuous coefficients. To overcome this last feature, let us approximate the problem as the following. In the case of problem (6), let us define

$$\begin{aligned} \tilde{f}_\varepsilon(x, u, t) &= \mathbb{I}_{\{u > 0\}} u \left[ \mathbb{I}_{\{x_{n+1} < C - \varepsilon\}} + \varepsilon^{-1} (x_{n+1} - C + \varepsilon) \mathbb{I}_{\{x_{n+1} \in [C - \varepsilon, C]\}} \right] + \mathbb{I}_{\{u \leq 0\}} u, \\ \tilde{h}_\varepsilon(x, u, t) &= (e^{x_1} - u) e^{x_2} \left[ \mathbb{I}_{\{x_{n+1} > \varepsilon\}} + \varepsilon^{-1} x_{n+1} \mathbb{I}_{\{x_{n+1} \in [0, \varepsilon]\}} \right]. \end{aligned}$$

In the case of problem (14), let us define  $\tilde{f}_\varepsilon = (\tilde{f}_{\varepsilon,1}, \dots, \tilde{f}_{\varepsilon,2m})$  and  $\tilde{h}_\varepsilon$  as

$$\begin{aligned} \tilde{f}_{\varepsilon,i}(x, u, t) &= \mathbb{I}_{\{u_i > 0\}} u \left[ \mathbb{I}_{\{x_i < C - \varepsilon\}} + \varepsilon^{-1} (x_{n+i} - C + \varepsilon) \mathbb{I}_{\{x_{n+i} \in [C - \varepsilon, C]\}} \right] + \mathbb{I}_{\{u_i \leq 0\}} u_i, \\ \tilde{h}_\varepsilon(x, u, t) &= (e^{x_1} - \sum_{i=1}^m u) e^{x_2} \prod_{i=1}^m \left[ \mathbb{I}_{\{x_{n+i} > \varepsilon\}} + \varepsilon^{-1} x_{n+i} \mathbb{I}_{\{x_{n+i} \in [0, \varepsilon]\}} \right] \\ &\quad + \phi(\{x_{n+i}\}_{i=1}^m, \{x_{n+m+i}\}_{i=1}^m, u, t). \end{aligned}$$

Let functions  $f_\varepsilon = (f_{\varepsilon,1}, \dots, f_{\varepsilon,2m}, h_\varepsilon, \text{ and } \Phi_\varepsilon)$ , be obtained via convolutions in  $x$  of the functions  $\tilde{f}(x, u, t)$ ,  $\min(\tilde{h}_\varepsilon(x, u, t), \varepsilon^{-1})$ , and  $\min(\Phi(x), \varepsilon^{-1})$ , respectively, with appropriate smooth convolution kernel  $\bar{k}_\varepsilon(x)$  such as  $\bar{k}_\varepsilon(x) = \varepsilon^{-1}\bar{k}(x/\varepsilon)$ , where  $\bar{k}(x)$  is a smooth enough kernels with finite support that is vanishing as  $\varepsilon \rightarrow 0$ ; see e.g. Krylov [22], pp. 48–49. We assume that  $f_\varepsilon = (f_{\varepsilon,1}, \dots, f_{\varepsilon,2m})$  for the case of problem (14).

Let  $g_\varepsilon$  be defined similarly to  $g$  with  $f$  replaced by  $f_\varepsilon$ .

The following holds:

- (i) The functions  $f_\varepsilon(x, u, t)$ ,  $h_\varepsilon(x, u, t)$ , and  $\Phi_\varepsilon(x)$ , are bounded and continuously differentiable in  $x \in \mathbf{R}$  for all  $u \leq e^{x_1}$ . The corresponding derivatives are bounded for all  $\varepsilon > 0$ .
- (ii)  $f_{\varepsilon,i}(x, u, t) \leq f_i(x, u, t)$ ,  $h_\varepsilon(x, u, t) \leq h(x, u, t)$ , and  $\Phi_\varepsilon(x) \leq \Phi(x, u, t)$  for all  $x, t$  and  $u \leq e^{x_1}$ .
- (iii)  $f_\varepsilon(x, u, t) \rightarrow f(x, u, t)$ ,  $h_\varepsilon(x, u, t) \rightarrow h(x, u, t)$ ,  $\Phi_\varepsilon(x) \rightarrow \Phi(x)$  as  $\varepsilon \rightarrow 0$  for all  $x, t$  and  $u \leq e^{x_1}$ .

Consider the following stochastic control problem:

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E} \left[ \int_0^T h_\varepsilon(x(t), u(t), t) dt + \Phi_\varepsilon(x(T)) \right] \quad \text{over } u(\cdot) \in \mathcal{U}_C \\ \text{subject to} \quad & dx(t) = g_\varepsilon(x(t), u(t), t) dt + \beta(x(t), t) dw(t), \quad x(0) = x_0. \end{aligned} \quad (15)$$

For  $\varepsilon \geq 0$ , consider the corresponding value function

$$J_\varepsilon(x, t) \triangleq \sup_{u(\cdot) \in \mathcal{U}_C} \mathbf{E} \left\{ \int_0^T h_\varepsilon(x(t), u(t), t) dt + \Phi_\varepsilon(x(T)) \middle| x(t) = x \right\}. \quad (16)$$

In particular,  $J_0(t, y_0)$  is the optimal value function for problem (12); by Theorem 3.1, this is also the optimal value function for problem (13) and problem (14).

Let  $D = \mathbf{R}^{n+1} \times [0, T]$  for problem(6), and let  $D = \mathbf{R}^{n+2m} \times [0, T]$  for problem (14). Let  $\mathcal{V}$  be the class of continuous functions  $v(x, t) : D \rightarrow \mathbf{R}$  such that there exists  $c > 0$  such that  $|v(x, t)| \leq c(|x| + 1)$  for all  $(x, t) \in D$ . Let  $\mathcal{V}_1$  be the class of functions  $v \in \mathcal{X}$  such that  $v'_t \in \mathcal{V}$  and that all components of  $v'_x$  belong to  $\mathcal{V}$ .

**Theorem 4.1** *For  $\varepsilon > 0$ , the value function  $J = J_\varepsilon$  is a solution of the following special Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} J'_t + \max_{u \in [-L, \min(\exp(x_1), L)]} \{J'_x g_\varepsilon + h_\varepsilon\} + \frac{1}{2} \text{Tr} \left( \beta^\top J''_{xx} \beta \right) &= 0, \\ J(x, T) &= \Phi_\varepsilon(x). \end{aligned} \quad (17)$$

Moreover, this boundary value problem has unique solution in the class of functions  $J = J_\varepsilon \in \mathcal{V}_1$ . The HJB equation holds as an equality that is satisfied for a.e.  $(x, t) \in D$ .

The following theorem establishes the way to approximate the solution  $J_0$  of the original problem.

**Theorem 4.2** *The solution  $J_0$  of the original problem can be approximated as*

$$J_0(x_0, 0) = \lim_{\varepsilon \rightarrow 0} J_\varepsilon(x_0, 0). \quad (18)$$

**Remark 4.1** In the present paper, the case where  $(p(t), S(t))$  is a part of stochastic Markov diffusion process was considered similarly to [13]. A more general case of right-continuous processes  $(p, S)$  can be considered similarly to Bender and Dokuchaev [5, 6], but this would require more complicated analysis. It can be also noted that the continuous time model could be replaced by a similar discrete time model.

Solution of Hamilton-Jacobi-Bellman equation (17) can be obtained via backward calculation after discretisation and transition to finite differences; see examples in [7] and [5]. However, for a large  $n + 2m$ , numerical implementation will be challenging.

The dimension of the HJB equation is defined by the number of factors captured by the model. If  $\phi \equiv 0$ ,  $p(t)$  is non-random and  $\log S(t)$  is a Brownian motion, then we can select  $n = 1$  and  $m = 0$ . In this case, the state space will be two dimensional (for brevity, the case of non-random  $p$  was excluded in the model described above). If  $m = 1$  (i.e. we consider one battery only), and if  $(\log p(t), \log S)$  is a 2D Brownian motion, then  $n = 2$ . In this case, the dimension of the state space is  $n + 2m + 1 = 5$ . Note that modelling of energy prices and the production rate is a non-trivial task and may require rather a large number of factors [16].

We address some possible ways to overcome the problem of high dimension in Section 5 below.

## 5 Duality and pathwise optimization for the value function

In this section, we assume that  $\mathcal{G}_t \subset \mathcal{F}_t$ , where  $\{\mathcal{F}_t\}$  is the filtration generated by a Wiener process  $W(t)$  taking the values in  $\mathbf{R}^m$ . We assume that  $X(t)$  is a stochastic process of a quite general type on a probability space  $(\Omega, P, \mathcal{F}_T)$ .

Consider a linear normed space  $\bar{\mathcal{X}} = L_2([0, T]; L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^m))$ . Let  $\mathcal{X}$  be the closed subspace obtained as the closure of the set of all progressively measurable with respect to  $\{\mathcal{F}_t\}$  processes from  $\bar{\mathcal{X}}$ .

Consider a set  $\bar{U} \subset \bar{\mathcal{X}}$  of control processes  $u(t, \omega)$  with the values at  $\mathbf{R}^m$ ; we assume that  $\bar{U}$  is convex and closed in  $\bar{\mathcal{X}}$ . Let  $U = \bar{U} \cap \mathcal{X}$ .

Assume that we are given a continuous concave function  $G : U \rightarrow \mathbf{R}$ . Let consider an optimal control problem

$$\text{Maximize } \mathbf{E}G(u) \quad \text{over } u \in U. \quad (19)$$

Let  $y_u^{(0)} = u$ ,  $y_u^k(t) = \int_0^t y_u^{(k-1)}(s)ds$ ,  $k = 1, 2, 3, \dots$  where  $y_u(t) = \int_0^t u(s)ds$ .

Consider a linear normed space  $\bar{V} = L_2([0, T]; L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^{m \times m}))$ . Let  $V$  be the closed subspace obtained as the closure of the set of all progressively measurable with respect to  $\{\mathcal{F}_t\}$  processes from  $\bar{V}$ .

**Lemma 5.1** *Let  $u \in \bar{U}$ . The following statements are equivalent:*

- (i)  $u \in U$ .
- (ii) There exists  $k > 1$  such that  $y_u^{(k)} \in \mathcal{X}$ .
- (iii) For any  $k \geq 0$ ,  $y_u^{(k)} \in \mathcal{X}$ .
- (iv) For any  $k \geq 0$  and any  $v \in V$ ,

$$\mathbf{E} \int_0^T \mu(t)^\top y_u^{(k)}(t) dt = \mathbf{E} \int_0^T \mu^{(k)}(t)^\top u(t) dt = 0.$$

Here  $M(t) = \int_0^t v(s)dW(s)$ ,  $\mu(t) = M(T) - M(t)$ ,  $\mu^{(0)} = \mu$ ,  $\mu^{(k)}(t) = -\int_t^T \mu^{(k-1)}(s)ds$ ,  $k = 1, 2, 3, \dots$

Let  $k \in \{0, 1, 2, \dots\}$  be selected.

For  $v \in V$ , let  $M(t) = \int_0^t v(s)dW(s)$ . Clearly,  $M(T) \in L_2(\Omega, \mathcal{F}_T, \mathbf{P})$  and  $M(t) = \mathbf{E}_t M(T)$ . Set  $\mu(t) = M(T) - M(t)$ . For  $u \in \bar{U}$ ,  $v \in V$ , and  $\mu^{(k)} = \mu^{(k)}(\cdot, v)$  defined as above, introduce Lagrangian

$$\mathcal{L}(u, v) = G(u) + \mathbf{E} \int_0^T \mu^{(k)}(t)^\top u(t) dt.$$

### Theorem 5.1

$$\sup_{u \in U} \mathbf{E}G(u) = \sup_{u \in \bar{U}} \inf_{v \in V} \mathcal{L}(u, v) = \inf_{v \in V} \sup_{u \in \bar{U}} \mathcal{L}(u, v). \quad (20)$$

It can be noted that Theorem 5.1 does not establish the existence of a saddle point. Hence this theorem does not give a way to derive an optimal strategy  $u$ . However, it can be used to estimate the value  $\sup_{u \in \bar{U}} \mathbf{E}G(u)$  using Monte-Carlo simulation of  $\mu^{(k)}$  and pathwise solution of the problem  $\sup_{u \in \bar{U}} \mathcal{L}(u, v)$  in the spirit of the methods developed in [2, 3, 4, 9, 10, 11, 15, 26]; this supremum can be found using pathwise optimization in the class of *anticipating* controls  $u \in \bar{U}$  that do not have to be adapted. An advantage of this approach is that it seeks only the solution starting from a particular  $y(0)$ , whereas the HJB approach described above requires to calculate the solution from all starting points. Overall, Theorem 5.1 gives an opportunity to estimate how far from optimal is the performance of a particular strategy, for instance, a strategy suggested in [18].

The papers [2, 3, 4, 9, 10, 11, 15, 26] mentioned here suggest to run Monte-Carlo over a set of martingales that are considered to be independent variables for the Lagrangian. In the term of Theorem 5.1, this means maximization over the set  $v \in V$ . Unfortunately, this set is quite wide. On the other hand, the optimal martingale have a very particular dependence on the underlying stochastic process and optimal value function, in the cases of some known explicit solutions. For example, for a related problem considered in [6], the corresponding optimal martingale was  $\mu(t) = J'_y(t, Y(t))$ , where  $Y(t)$  was an optimal state process,  $J(t, y)$  was the optimal value function for the problem satisfying a backward stochastic first order Hamilton–Jacobi–Bellman equation (Theorem 5.1 [6]). For a discrete time setting, a similar representation for an optimal martingale was obtained in [26]. This shows that a sequence of randomly generated martingales may not attend a close proximity of the optimal martingale in a reasonable time. The novelty of Theorem 5.1 is that it allows to replace simulation of martingales by simulation of more special processes  $\mu^{(k)}(t)$ . In particular, processes  $\mu^{(k)}(t)$  are  $k - 1$  times pathwise differentiable, with absolutely continuous derivative  $d^{k-1}\mu^{(k)}(t)/dt^{k-1}$ . After time discretisation, these processes can be described as processes with a reduced range of finite differences of order  $k$ . This could help to reduce the calculation time. The case where  $k = 0$  corresponds to martingale duality studied in [2, 3, 4, 6, 9, 10, 11, 15, 26].

In particular, Theorem 5.1 is applicable to problem (12) for the case where the function

$\phi(y, \bar{y}, u, t)$  is concave in  $(y, \bar{y}) \in \mathbf{R}^m \times \mathbf{R}^m$ . To apply Theorem 5.1 for problem (12), one have to select

$$\begin{aligned} \bar{U} = \{u \in L_2([0, T]; L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^m)) : \\ u_i(t) \in [-L, \min(p(t), L)], \quad y_i(0) + \int_0^t u_i(s)ds \in [0, C] \quad \text{a.e.,} \quad i = 1, \dots, m\}. \end{aligned}$$

## 6 Proofs

*Proof of theorem 4.1.* The coefficients of the problem are such that the assumptions of Theorem 4.1.4 and Theorem 4.4.3 from Krylov [22], pp. 167 and 192, are satisfied.

By Theorem 4.1.4 from [22], p. 167, the function  $J$  satisfies the corresponding parabolic Bellman equation such that it has unique solution in  $\mathcal{V}$  and all components of  $J'_x$  belong to  $\mathcal{V}$ . The Bellman equation holds in the generalized sense, i.e. as an equality of the distributions. By Theorem 4.4.3 from [22], p.192, the derivative  $J'_t(x, t)$  belongs to  $\mathcal{V}$ . Then the proof of theorem 4.1 follows.  $\square$ .

*Proof of Theorem 4.2.* Let  $J_C$  be defined as  $J$  with a particular selection of  $C$ . Let us consider a set of variable  $C_1$  and  $C_2$  such that  $0 < C_1 < C_2 \leq \text{const}$ . Clearly,  $J_{C_1}(x, 0) \leq J_{C_2}(x, 0)$ . Let us show that

$$J_{C_1}(x, 0) - J_{C_2}(x, 0) \rightarrow 0 \quad \text{as} \quad C_2 - C_1 \rightarrow 0. \quad (21)$$

We have that  $J_{C_2}(y_0, 0) \geq \mathbf{E}F(\tilde{u}_k) - 1/k$  for some  $\tilde{u}_k \in \mathcal{U}_{C_2}$ . Let

$$\tilde{y}_k(t) = y_0 + \int_0^t \tilde{u}_k(s)ds.$$

Without a loss of generality, we assume that  $\tilde{y}_{k,i}(t) \leq C_2$ ,  $i = 1, \dots, m$ , for the components of  $\tilde{y}_k(t) = (\tilde{y}_{k,1}(t), \dots, \tilde{y}_{k,m}(t))$ . Let  $\hat{u}_k(t) = (\hat{u}_{k,1}(t), \dots, \hat{u}_{k,m}(t))$  be defined as  $\hat{u}_{k,i}(t) = \tilde{u}_{k,i}(t)$  if  $\tilde{y}_{k,i}(t) < C_1$ , and  $\hat{u}_{k,i}(t) = 0$  if  $\tilde{y}_{k,i}(t) \geq C_1$ . Let

$$F_C(u) = \int_0^T h(x_u(t), u(t), t)dt + \Phi(X(T)), \quad (22)$$

where  $x_u(\cdot)$  is defined by (6) given  $x(0) = x_0$ . Clearly,  $J_{C_1}(x, 0) \geq \mathbf{E}F_{C_1}(\hat{u}_k)$  and

$$\mathbf{E}F_{C_1}(\hat{u}_k) - \mathbf{E}F_{C_2}(\hat{u}_k) \rightarrow 0 \quad \text{as} \quad C_2 - C_1 \rightarrow 0.$$



Then (21) holds.

Further, let  $J_{C,\varepsilon}$  be defined as  $J_\varepsilon$  given a particular selection of  $C$ . It follows from the definitions that

$$J_{C-\varepsilon,\varepsilon/2}(x, 0) \in [J_{C-\varepsilon}(x, 0), J_{C+\varepsilon}(x, 0)].$$

Then the proof follows.  $\square$

*Proof of Lemma 5.1* is straightforward and will be omitted here. It is based on the fact that, by the Martingale Representation Theorem,  $\mu(t) = \int_t^T \hat{\mu}(s)dw(s)$ , for some  $\hat{\mu} \in \mathcal{V}$ . for  $\square$

*Proof of Theorem 5.1.* The first equality follows from Lemma 5.1. Furthermore, we have that  $\mathcal{L}(u, v)$  is concave in  $u \in U$  and affine in  $v \in V$ . In addition,  $\mathcal{L}(u, v)$  is continuous in  $u \in L_2([0, T] \times \Omega)$  given  $v \in V$ , and  $\mathcal{L}(u, v)$  is continuous in  $v \in V$  given  $u \in U$ . The statement of the theorem follows Proposition 2.3, from [14], Chapter VI. Statement (ii) follows from (i) and Proposition 1.2 from [14], Chapter VI. This completes the proof of Theorem 5.1.  $\square$

The paper suggest a compact and yet comprehensive model for decision making under uncertainty for a of small or medium producer of energy. As a result, a method of calculation of optimal storing and dispatching strategy is suggested. This method allows to increase the income for a grid-connected energy producer with battery energy

## 7 Conclusion

The paper suggest a compact and yet comprehensive model for decision making under uncertainty for a of small or medium producer of energy. As a result, a method of obtaining an optimal operating algorithm for storing and dispatching strategy is suggested. This method allows to increase the income for a grid-connected energy producer with battery energy storage system (BESS). In addition, the suggested method allows to take into account preferable regimes for batteries charging and discharging, to prolong the battery life. This feature is new and was not investigated in the existence literature. However, this addition is essential for decision making given that the batteries are expensive.

The model developed is a stochastic model meaning that the underlying processes are assumed to be random with some given probability distributions. The part of the model describing the storing and dispatching the energy is still valid without this probabilistic assumption. Moreover,

stochastic type uncertainty can be replaced by interval type uncertainty, and expectation in the optimality criterion can be replaced by maximin type criterion. However, the numerical methods considered in the present paper rely on the stochastic model of uncertainty.

Usually, decision making in the stochastic framework relies on forecasting the underlying processes. In the model described above, the forecasting is assumed in an implicit form: we assume an evolution model (4)-(5) for the production rate and prices, and this model implies optimal forecasts for the underlying processes as their conditional expectations given the observations. This forecast and the corresponding error can be calculated analytically from equations (4)-(5). One has to apply statistical inference and forecasting methods to calibrate the parameters for these equations. However, this requires special consideration beyond the scope of this paper. We leave this for the future research. An example of a forecasting method applicable for wind energy production can be found in [19].

It has to be emphasized that the model introduced in this paper is quite flexible and allow many modifications that were not executed in the present paper to avoid overloading by technical details. For example, the following straightforward modifications are possible:

- (i) The capacity of the batteries can be allowed to be selected individually for all batteries; in the current version, the capacity is presumed to be the same for all batteries.
- (ii) The performance criterion could be selected to ensure risk aversion, i.e. impose higher penalty on losses than rewards for the gain.
- (iii) In the present form, the model describes processes as continuous time processes. However, this is rather technical assumption since a similar discrete time model can be obtained via straightforward discretisation.
- (iv) The decision making model does not require a particular model for the dynamics of the production rate and the market price processes. Any other model of dynamics of the production rate and the market prices can be accommodated instead of the diffusion model described in Section 3.

Numerical implementation via discretization of the suggested method is straightforward and requires backward solution of the dynamic programming equation. However, the state space

dimension for the equation for the optimal value function could be high for a large number of factors arising in a multi-battery setting. This means that numerical solution could require significant computational efforts. To address this, we suggested a method based on duality and pathwise optimization based on the approach developed in [2, 3, 4, 9, 10, 11, 15, 26]. We leave further development of the numerical methods for the future research.

Several other important factors have been left for further future research. An example is taking into account possible cooperation with the grid operators and other producers.

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